

A Frobenius-Schur theorem for Hopf algebras

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To Klaus Roggenkamp on his 60th birthday

1 Introduction

In this note we prove a generalization of the Frobenius-Schur theorem for finite groups for the case of semisimple Hopf algebra over an algebraically closed field of characteristic 0. A similar result holds in characteristic $p > 2$ if the Hopf algebra is also cosemisimple. In fact we show a more general version for any finite-dimensional semisimple algebra with an involution; this more general result (and its proof) may give some new insight into the classical theorem.

Let G be a finite group. For $h \in G$, define $\vartheta_m(h)$ to be the number of solutions of the equation $g^m = h$, that is $\vartheta_m(h) = |\{g \in G \mid g^m = h\}|$.

Because $\vartheta_m(h)$ is a class function it can be written as

$$\vartheta_m(h) = \sum_{\chi \in Irr(G)} \nu_m(\chi) \chi(h) \quad (1)$$

where $Irr(G)$ is the set of irreducible characters of G . By the orthogonality relations for characters one can prove that

$$\nu_m(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^m).$$

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The classical Frobenius-Schur theorem for groups describes properties of the coefficients $\nu_2(\chi)$, for $\chi \in \text{Irr}(G)$, and of the corresponding irreducible representations V_χ . In particular it says that the irreducible representations fall into three classes.

We can now state the classical theorem; for a reference see [Se] or [I].

Frobenius-Schur Theorem *Let $k = \mathbf{C}$, let G be a finite group, and let $\nu_2(\chi)$ be as above. Then*

- (1) $\nu_2(\chi) = 0, 1$ or -1 , for all $\chi \in \text{Irr}(G)$;
- (2) $\nu_2(\chi) \neq 0$ if and only if χ is real-valued (equivalently $\chi(g) = \chi(g^{-1})$ for all $g \in G$). Moreover $\nu_2(\chi) = 1$ (respectively -1) if and only if V_χ admits a symmetric (resp. skew-symmetric) non-degenerate bilinear G -invariant form.
- (3) $1 + t = \sum_{\chi \in \text{Irr}(G)} \nu_2(\chi) \chi(e_G)$, where t is the number of elements of order two in G ;
- (4) $\chi^{(2)}(g) := \chi(g^2)$ is a difference of two characters.

$\nu_2(\chi)$ is called the *Schur indicator* of χ . It is also well-known that over a field k of characteristic $p \neq 0, 2$, with $|G|$ relatively prime to p , the representation theory of G is equivalent to its representation theory in characteristic 0. Thus a similar result holds in characteristic $p \neq 2$ [Se].

Now let H be a finite-dimensional semisimple Hopf algebra over k , with comultiplication Δ , counit ε , and antipode S . We will show that the analog of properties (1), (2), and (3) can be proved for H . First we define the analog of ν for H . The power g^m is replaced by the “generalized power map” for Hopf algebras; that is, for any $h \in H$,

$$h^{[m]} := \sum_{(h)} h_1 h_2 \cdots h_m,$$

where $\Delta_{m-1}(h) = \sum_{(h)} h_1 \otimes \cdots \otimes h_m$. Note that for $g \in G$, $g^m = g^{[m]}$. This power map was studied classically and has recently been a renewed object of interest [K1], [EG2]. Since H is semisimple we may choose $\Lambda \in \int_H$ with $\varepsilon(\Lambda) = 1$. Then Λ replaces $\frac{1}{|G|} \sum_{g \in G} g$. Thus for any m we define

$$\nu_m(\chi) := \sum_{(\Lambda)} \chi(\Lambda_1 \cdots \Lambda_m).$$

Next, if V is any left H -module, then V^* is also a left H -module using S : if $f \in V^*$, $v \in V$, and $h \in H$, then $(h \cdot f)(v) := f(Sh \cdot v)$. We write χ_V

for the character corresponding to V , and for a character χ of H , we write V_χ for the H -module corresponding to χ . Note that $\chi_{V^*} = \chi_V \circ S$.

Finally, we say a bilinear form $(\ , \)$ on V is H -invariant if for any $h \in H$ and any $v, u \in V$ we have

$$h \cdot (v, w) = \sum_{(h)} (h_1 \cdot v, h_2 \cdot u) = \varepsilon(h)(v, u).$$

The next result, our main theorem, will be proved in Section 3.

Theorem 3.1. *Let H be a semisimple Hopf algebra over an algebraically closed field k . If k has characteristic $p \neq 0$, assume in addition that $p \neq 2$ and that H^* is semisimple. Then for Λ and $\nu_2(\chi)$ as above, and $\chi \in \text{Irr}(H)$, the following properties hold:*

- (1) $\nu_2(\chi) = 0, 1 \text{ or } -1, \forall \chi \in \text{Irr}(H)$,
- (2) $\nu_2(\chi) \neq 0$ if and only if $V_\chi \cong V_\chi^*$. Moreover $\nu_2(\chi) = 1$ (respectively -1) if and only if V_χ admits a symmetric (resp. skewsymmetric) non-degenerate bilinear H -invariant form.
- (3) Considering $S \in \text{End}(V)$, $\text{Tr} S = \sum_{\chi \in \text{Irr}(H)} \nu_2(\chi) \chi(1_H)$.

As for groups, we will call $\nu_2(\chi)$ the *Schur indicator* of χ .

We see that this result does indeed generalize the classical theorem. (1) is the same, and for (2), recall that for groups over \mathbf{C} , a character is real-valued if and only if the corresponding module is self-dual. However we do not know how to formulate the exact analog of a character being real-valued in the Hopf algebra situation, since in general we do not have a canonical basis of H which plays the role of the group elements in the group algebra.

Finally part (3) of the theorem becomes the formula for the number of involutions in the group algebra case: since in this case the antipode is given by $S(g) = g^{-1}$, the matrix of S computed with respect to the basis of group elements has non-zero diagonal entries only for those elements such that $g = g^{-1}$. Thus the trace of S is precisely $1 + t$, where t is the number of involutions of G .

One might hope to prove the analog of (4), namely that $\chi^{(2)}(h) := \sum_{(h)} \chi(h_{(1)} h_{(2)})$, as a function of h , is a difference of two characters. However this is false in general; we will see a counterexample in Section 3.

For general references on Hopf algebras, we refer to [M] and [Sch].

2 Algebras with Involution.

In this section A is an arbitrary split semisimple algebra with an involution S over k . That is, S is an antiautomorphism of order 2 and A is a direct sum of full matrix rings over k , say $A = \bigoplus_{i=1}^d M_{n_i}(k)$. Let $\{e_i\}_{i=1}^d$ be the set of primitive central idempotents of A ; then S permutes the $\{e_i\}$. For each i , let V_i be the corresponding irreducible left A -module with character χ_i . We also denote χ_i by tr_i , the usual matrix trace in the i th component. For any left A -module W , W^* is also a left A -module, using S as was done in Section 1. That is, for $f \in W^*$, $a \in A$, $w \in W$, define $(a \cdot f)(w) := f(S(a) \cdot w)$.

Lemma 2.1. *If $S(M_{n_i}(k)) = M_{n_j}(k)$, then $V_j \cong V_i^*$ as A -modules.*

Proof. Since $M_{n_i}(k) \cong V_i^{(n_i)}$ as A -modules, it suffices to show that $M_{n_j}(k) \cong (M_{n_i}(k))^*$ as (left) A -modules. To see this define

$$\Phi : M_{n_j}(k) \longrightarrow M_{n_i}(k)^*$$

via $\Phi(a) = a \cdot \chi_i$, where \cdot is the action above using $W = M_{n_i}(k)$. It is easy to see that Φ is an A -module map. It is injective (and so bijective) by the non-degeneracy of the trace. Thus

$$V_j \cong V_i^*.$$

□

We will denote by $*$ the permutation on $\{1, \dots, d\}$ induced by S . Thus $S(M_{n_i}(k)) = M_{n_{i^*}}(k)$. We now wish to consider TrS , the trace of S considered as an element of $End(A)$. To do this we consider the trace of S on each of the $M_{n_i}(k)$.

Lemma 2.2. *If $S(M_{n_i}(k)) \neq M_{n_i}(k)$, or equivalently if V_i is not self-dual, then $TrS|_{M_{n_i}(k) \oplus M_{n_{i^*}}(k)} = 0$.*

Proof. If B_i is a basis for $M_{n_i}(k)$ and B_{i^*} is a basis for $M_{n_{i^*}}(k)$, then $B_i \cup B_{i^*}$ is a basis for $M_{n_i}(k) \oplus M_{n_{i^*}}(k)$. Restricted to this subalgebra S has matrix of the form

$$\begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix}$$

and so $TrS|_{M_{n_i}(k) \oplus M_{n_i^*}(k)} = 0$. That this is the case when V_i is not self-dual is just Lemma 2.1. \square

We now consider the self-dual case, that is $S(M_{n_i}(k)) = M_{n_i}(k)$, or equivalently $V_i^* \cong V_i$. We let $(\)^t$ denote the transpose map on $M_{n_i}(k)$,

We use the following result, due to A. A. Albert, which describes the possible involutions on a matrix algebra. A modern reference is [J, Sec 5.1].

Theorem 2.3. *Let $A = End_k(V) \cong M_n(k)$, where V is an n -dimensional vector space over k , and k has characteristic $\neq 2$. Assume that A has a k -involution S . Then there exists a non-degenerate bilinear form $(\ , \)$ on V such that S is the adjoint with respect to the form; that is, $(av, w) = (v, S(a)w)$ for all $v, w \in V$, $a \in A$. Moreover the form is either symmetric or skew-symmetric. In each case S can be described more precisely:*

1. *The symmetric case. In this case, one may choose a basis of V so that considering A as matrices with respect to this basis,*

$$S(a) = Da^tD^{-1}$$

for all $a \in A$, where $D = diag\{d_1, \dots, d_n\}$, a diagonal matrix.

2. *The skew case. In this case $n = 2m$, and one may choose a basis of V so that considering A as matrices with respect to this basis,*

$$S(a) = Ga^tG^{-1}$$

for all $a \in A$, where $G = diag\{C_1, \dots, C_m\}$ and each $C_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, a 2×2 block.

Corollary 2.4. *Let $A = M_n(k)$ and let S be a k -involution of A . Consider the two possibilities for S in Albert's theorem. Then in case (1), $TrS = n$, and in case (2), $TrS = -n$.*

Proof. In both cases, choose a basis for A of matrix units $\{e_{ij}\}$ with respect to the special bases of V in Theorem 2.3.

In case (1), $S(e_{ij}) = d_i d_j^{-1} e_{ji}$. Thus the only non-zero contributions to TrS come from $S(e_{ii}) = e_{ii}$, for $i = 1, \dots, n$, and so $TrS = n$.

In case (2), $S(e_{ij}) \in ke_{ij}$ only if e_{ij} lies in a 2×2 diagonal block. It suffices to consider the upper left such block. Conjugating by C_1 , we see

$$S \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \begin{pmatrix} \alpha_{22} & -\alpha_{12} \\ -\alpha_{21} & \alpha_{11} \end{pmatrix}.$$

That is, $S(e_{11}) = e_{22}$, $S(e_{12}) = -e_{12}$, $S(e_{21}) = -e_{21}$, and $S(e_{22}) = e_{11}$. Thus $TrS = -2$ on this block, and so altogether $TrS = -2m = n$ since there are m blocks. \square

Corollary 2.5 *Let A and S be as in Corollary 2.4 and let $\{e_{ij}\}$ be a basis of matrix units for A with respect to the bases in Theorem 2.3. Let tr be the usual matrix trace in A . Then:*

- (1) *In the symmetric case, $\sum_{l,m=1}^n tr(S(e_{lm})e_{ml}) = n$.*
- (2) *In the skew case, $\sum_{l,m=1}^n tr(S(e_{lm})e_{ml}) = -n$.*

Proof. One can show this directly, similarly to the proof of 2.4, by using the formula for S given in 2.3.

Alternatively it follows from 2.4, since for any map $U \in End_k(A)$, one may verify that $Tr(U) = \sum_{l,m} tr(U(e_{lm})e_{ml})$. We thank H.-J. Schneider for pointing out to us this second proof. \square

We now return to the general case when A is split semisimple. Suppose $\langle | \rangle$ is a bilinear, associative, symmetric, non-degenerate form on A . Since the only linear functional f on $M_n(k)$ such that $f(ab) = f(ba)$, all $a, b \in M_n(k)$, is the trace (up to a scalar), it follows that for some non-zero scalars $\gamma_i \in k$,

$$\langle a|b \rangle = \sum_{i=1}^d \gamma_i tr_i(ab).$$

The set $\{a_r, b_r\}$, for $r = 1, \dots, \dim A$, is called a pair of *dual bases* for A with respect to this form if $\langle a_r|b_j \rangle = \delta_{rj}$, for all r, j . If $\{a_r, b_r\}$ is a pair of dual bases, then for all $c \in A$,

$$c = \sum_j \langle a_j|c \rangle b_j = \sum_j \langle c|b_j \rangle a_j.$$

For example, for $\langle | \rangle$ as above, one pair of dual bases is given by $\{\gamma_i^{-1}e_{lm}^i, e_{ml}^i\}$, where for each i , $\{e_{lm}^i\}$ is a set of matrix units for the i th summand $M_{n_i}(k)$.

The next lemma is well-known.

Lemma 2.6. *Let V be a finite-dimensional vector space with a non-degenerate form $\langle | \rangle$ on V . Assume that $\{a_r, b_r\}$ and $\{c_j, d_j\}$ are two pairs of dual bases for V with respect to the form. Then*

$$\sum_r a_r \otimes b_r = \sum_j c_j \otimes d_j.$$

Proof. Consider the map $\Phi : V \otimes V \longrightarrow V^* \otimes V^*$ defined by $\Phi(v \otimes w) = \langle v | - \rangle \otimes \langle - | w \rangle$. Then Φ is an isomorphism of vector spaces, because $\langle - | - \rangle$ is non-degenerate. Thus to see that the two sums are equal, it suffices to show that their images under Φ are equal.

However this follows by applying both images to $b_p \otimes a_q$, for all b_p, a_q , and using on the right the fact that $\sum_j \langle a_q | d_j \rangle c_j = a_q$. \square

We can now prove our main theorem on algebras with involution.

Theorem 2.7. *Let A be a finite-dimensional split semisimple algebra over k , and write $A = \bigoplus_{i=1}^d M_{n_i}(k)$ as above. Assume that k has characteristic $\neq 2$ and that each $n_i \neq 0$ in k . Assume that A has a k -involution S . Let V_1, \dots, V_d be the distinct irreducible modules for A and let χ_1, \dots, χ_d be the corresponding irreducible characters. Also let $\{a_r, b_r\}$ be a pair of dual bases with respect to some symmetric bilinear associative non-degenerate form $\langle | \rangle$ on A . Then the numbers*

$$\mu_2(\chi_i) := \frac{n_i}{\chi_i(\sum_j a_j b_j)} \chi_i(\sum_r S(a_r) b_r)$$

satisfy the following properties:

1. $\mu_2(\chi_i) = 0, 1$ or -1 , for all $\chi_i \in \text{Irr}(A)$.
2. $\mu_2(\chi_i) \neq 0$ if and only if $V_i \cong V_i^*$. Also $\mu_2(\chi_i) = 1$ (respectively -1) if and only if V_i admits a symmetric (resp. skew-symmetric) non-degenerate form such that $S|_{A_i}$ is the adjoint of the form, where A_i is the i th summand of A .
3. $\text{tr} S = \sum_{\chi \in \text{Irr}(A)} \mu_2(\chi) \chi(1_A)$.

Proof. By Lemma 2.6, we may replace the $\{a_r, b_r\}$, all r , in (1) by $\{\gamma_i^{-1}e_{lm}^i, e_{ml}^i\}$, all i, l, m . Now if $i^* \neq i$, then $S(e_{lm}^i) \notin A_i$, and so $\chi_i(\sum_{l,m} S(e_{lm}^i)e_{ml}^i) = 0$. Thus $\mu_2(\chi_i) = 0$ in this case.

Thus we may assume that $i^* = i$. Using the special dual bases above,

$$\chi_i(\sum_r S(a_r)b_r) = \sum_{l,m} \gamma_i^{-1} \text{tr}_i(S(e_{lm}^i)e_{ml}^i).$$

Apply Corollary 2.5 to see that in the symmetric case this equals $\gamma_i^{-1}n_i$ and in the skew case it equals $-\gamma_i^{-1}n_i$. One may also check directly that

$$\chi_i(\sum_r a_r b_r) = \sum_{l,m} \gamma_i^{-1} \text{tr}_i(e_{lm}^i e_{ml}^i) = \gamma_i^{-1} n_i^2.$$

Thus $\mu_2(\chi_i) = 1$ in the symmetric case, and similarly $\mu_2(\chi_i) = -1$ in the skew case. This proves (1) and (2).

To see (3), we use Lemma 2.2 and Corollary 2.4:

$$\text{Tr} S = \sum_i (\text{Tr} S|_{A_i}) = \sum_{i^*=i} \text{Tr}(S|_{A_i}) = \sum_{V_i \text{ symmetric}} n_i - \sum_{V_i \text{ skew}} n_i = \sum_i \mu_2(\chi_i) n_i.$$

But $n_i = \chi_i(1_A)$, the degree of χ_i .

□

3 Semisimple Hopf algebras

In this section we prove our main theorem on Hopf algebras, and give an example. Thus assume that H is a (finite-dimensional) semisimple Hopf algebra.

Proof of Theorem 3.1. If k has characteristic 0, then by [LR], $S^2 = id$; moreover they also prove that H^* is semisimple. If k has characteristic $p \neq 0$, then $S^2 = id$ provided H^* is also semisimple [EG1]. Thus in either case S is an involution on H . If characteristic $k = p$, then by [L, Theorem 2.8], the degree of each irreducible left H -module is relatively prime to p . Thus we may apply Theorem 2.7 to H .

We next show that $\mu_2(\chi) = \nu_2(\chi)$. Since both H and H^* are semisimple, they are unimodular, and so the spaces of left and right integrals coincide.

Now choose $\lambda \in \int_{H^*}$ such that $\lambda(1_H) = \dim H$ and $\Lambda \in \int_H$ with $\varepsilon(\Lambda) = 1$. Then by [L, Proposition 4.1], $\lambda(\Lambda) = 1$ and

$$\lambda = \sum_{\chi_i \in \text{Irr}(H)} n_i \chi_i.$$

Thus λ is the trace of the (left) regular representation of H .

Define a bilinear form $\langle | \rangle$ on H via

$$\langle a|b \rangle = \lambda(ab),$$

for all $a, b \in H$. It is clear that $\langle | \rangle$ is a non-degenerate associative symmetric bilinear form on H . It follows by [OS] that $\{S(\Lambda_1), \Lambda_2\}$ is a pair of dual bases with respect to $\langle | \rangle$. See also [Sch, Theorem 3.1].

Now in the bilinear form $\langle | \rangle$ above, $\gamma_i = n_i$. We have seen in the proof of Theorem 2.7 that it is always true that $\chi_i(\sum_j a_j b_j) = \gamma_i^{-1} n_i^2$; thus in our case $\chi_i(\sum_j a_j b_j) = n_i$. Using the dual bases above,

$$\mu_2(\chi_i) = \chi_i(\sum_j S(a_j) b_j) = \chi_i(\sum_{(\Lambda)} S^2(\Lambda_1) \Lambda_2) = \chi_i(\sum_{(\Lambda)} \Lambda_1 \Lambda_2) = \nu_2(\chi_i),$$

from the definition of $\nu_2(\chi)$ in Section 1.

Thus in order to finish the proof of Theorem 1 we only have to show that if $i = i^*$ then the bilinear form on V_i as in 2.7, part (2), is H -invariant. However this is trivial:

$$\sum_{(h)} (h_1 \cdot v, h_2 \cdot w) = \sum_{(h)} (v, S(h_1) h_2 \cdot w) = \varepsilon(h)(v, w)$$

since S is the adjoint map with respect to the form.

□

We now consider part (4) of the original Frobenius-Schur theorem. One would like to prove that $\chi^{(2)} = \sum_{(h)} \chi(h_1 h_2)$ is a difference of two characters. However this is false in general, as the following example shows.

Example 3.2. We use Example 15 of [K2]. Let $H = kQ_2 \#^\alpha kC_2$, the smash coproduct of the group algebras of the quaternion group Q_2 and the cyclic group C_2 of order 2. As an algebra, $H \cong kG$, the group algebra, where

$$G = Q_2 \times C_2 = \langle a, b, g \mid a^4 = e, b^2 = a^2, ba = a^{-1}b, ag = ga, bg = gb, g^2 = e \rangle.$$

The coalgebra structure of H is given explicitly in [K2] as follows:

$$\Delta(a) = \frac{1}{2}(a \otimes a + ag \otimes a + a \otimes b - ag \otimes b),$$

$$\Delta(b) = \frac{1}{2}(b \otimes b + bg \otimes b + b \otimes a - bg \otimes a),$$

$$\Delta(g) = g \otimes g, \quad S(g) = g,$$

$$S(a) = \frac{1}{2}(a^3 + a^3g + a^2b - a^2bg),$$

$$S(b) = \frac{1}{2}(b^3 + b^3g + a^3 - a^3g),$$

$$\varepsilon(a) = \varepsilon(b) = \varepsilon(g) = 1.$$

Since as an algebra, $H \cong k^{(8)} \oplus M_2(k)^{(2)}$, H has two irreducible characters of degree 2. These characters are described explicitly in [K2]. One can verify that for both of these characters, $\chi^{(2)}$ can not be expressed as a linear combination of characters.

We remark that this example was also studied by [N], who showed that its K_0 ring is non-commutative.

Finally we would like to note that we could not find any sensible interpretation for the function

$$\theta_m(h) := \sum_{\chi \in \text{Irr}(H)} \nu_m(\chi) \chi(h)$$

in the Hopf algebra case.

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